Goldstein

1.15. (a) In Cartesian Coordinates,

$$V(\vec{r},\vec{v}) = V(|\vec{r}|) + m\vec{\sigma}\cdot(\vec{r}\times\vec{v})$$

$$= V(|\vec{r}|) + m\vec{\sigma}\cdot\begin{bmatrix}(r_1 v_2 - r_2 v_1)\hat{z} + \\ (r_2 v_3 - r_3 v_2)\hat{x} + \\ (r_3 v_1 - r_1 v_3)\hat{y}\end{bmatrix}$$

$$= V(|\vec{r}|) + m\begin{bmatrix}\sigma_1(r_2\dot{r}_3 - r_3\dot{r}_2) + \\ \sigma_2(r_3\dot{r}_1 - r_1\dot{r}_3) + \\ \sigma_3(r_1\dot{r}_2 - r_2\dot{r}_1)\end{bmatrix}$$

$$\Rightarrow Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)$$

$$\frac{\partial U}{\partial r_1} = \frac{\partial V}{\partial|\vec{r}|}\frac{\partial|\vec{r}|}{\partial r_1} + m(\sigma_3\dot{r}_2 - \sigma_2\dot{r}_3)$$

$$\dot{\left(\frac{\partial U}{\partial \dot{r}_1}\right)} = m(\sigma_2\ddot{r}_3 - \sigma_3\ddot{r}_2)$$

$$Q_1 = -\frac{\partial V}{\partial|\vec{r}|}\frac{\partial|\vec{r}|}{\partial r_1} - m(\sigma_3\dot{r}_2 - \sigma_2\dot{r}_3) + m(\sigma_2\dot{r}_3 - \sigma_3\dot{r}_2)$$

$$= \boxed{-\frac{\partial V}{\partial|\vec{r}|}\frac{\partial|\vec{r}|}{\partial r_1} + 2m(\sigma_2\dot{r}_3 - \sigma_3\dot{r}_2)}.$$

Similarly, $Q_2 = -\frac{\partial V}{\partial|\vec{r}|}\frac{\partial|\vec{r}|}{\partial r_2} + 2m(\sigma_3\dot{r}_1 - \sigma_1\dot{r}_3)$

$$Q_3 = -\frac{\partial V}{\partial|\vec{r}|}\frac{\partial|\vec{r}|}{\partial r_3} + 2m(\sigma_1\dot{r}_2 - \sigma_2\dot{r}_1)$$

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1.15 (a). Now we consider the spherical case.

$\vec{\sigma} \cdot \vec{L} = |\vec{\sigma}||L| \cos\gamma$ where γ is the angle between the two vectors. This operation is invariant under rotation of both vectors, so WLOG let $\vec{\sigma} = (0, 0, |\sigma|)$.

Then $\vec{\sigma} \cdot \vec{L} = \vec{\sigma} \cdot (\vec{r} \times m\vec{v})$

$= m\vec{\sigma} \cdot (\vec{r} \times \vec{v})$

$$= m\vec{\sigma} \cdot \begin{bmatrix} (r_1 v_2 - r_2 v_1)\hat{z} + \\ (r_2 v_3 - r_3 v_2)\hat{x} + \\ (r_3 v_1 - r_1 v_3)\hat{y} \end{bmatrix}$$

$= m|\sigma|(r_1 \dot{r}_2 - r_2 \dot{r}_1)$.

$$= m|\sigma| \begin{bmatrix} r\sin\theta\cos\phi(\dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}) \\ -r\sin\theta\sin\phi(\dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}) \end{bmatrix}$$

$$= m|\sigma|\left[r^2\sin^2\theta\cos^2\phi\,\dot{\phi} + r^2\sin^2\theta\sin^2\phi\,\dot{\phi}\right]$$

$$= m|\sigma|\, r^2\sin^2\theta\,\dot{\phi}$$

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1.15 (a)

$$U(r,\vec{v}) = V(r) + m\vec{\sigma}\cdot L$$

$$= V(r) + m|\sigma| r^2 \sin^2\theta\, \dot{\phi}$$

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\frac{\partial U}{\partial r} = \frac{\partial V}{\partial r} + 2m|\sigma| r \sin^2\theta\, \dot{\phi}, \quad \frac{\partial U}{\partial \dot{r}} = 0.$$

$$\frac{\partial U}{\partial \theta} = m|\sigma| r^2 2\sin\theta\cos\theta\, \dot{\phi}, \quad \frac{\partial U}{\partial \dot{\theta}} = 0.$$

$$\frac{\partial U}{\partial \dot{\phi}} = m|\sigma| r^2 \sin^2\theta,$$

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\phi}}\right) = m|\sigma|\left[2r\dot{r}\sin^2\theta + r^2(2)\sin\theta\cos\theta\,\dot{\theta}\right]$$

$$\boxed{\begin{aligned} Q_r &= -\left(\frac{\partial V}{\partial r} + 2m|\sigma| r\sin^2\theta\,\dot{\phi}\right) \\ Q_\theta &= -m|\sigma| r^2 2\sin\theta\cos\theta\,\dot{\phi} \\ Q_\phi &= m|\sigma|\left[2r\dot{r}\sin^2\theta + 2r^2\sin\theta\cos\theta\,\dot{\theta}\right] \end{aligned}}$$

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